Date

MATH2230 Complex Variables with Application
Suggested Solution for HW8

Sect. 57 No. 1

Remark: When you use Cauchy integral formula, pay attention to the condition.

(a) $\int_C \frac{e^{-z}}{z-\pi i/2}\,dz = 2\pi i \cdot e^{-\pi i/2} = 2\pi$

(b) $\int_C \frac{\cos z}{z(z^2+8)}\,dz = 2\pi i \cdot \frac{\cos 0}{8} = \frac{\pi i}{4}$

(c) $\int_C \frac{z}{2z+1}\,dz = 2\pi i \cdot \frac{1}{2} \cdot (-\frac{1}{2}) = -\frac{1}{2}\pi i$

(d) $\int_C \frac{\cosh z}{z^4}\,dz = \frac{2\pi i}{3!} \cosh^{(3)}(z)\big|_{z=0} = \frac{2\pi i}{3!} \sinh 0 = 0$

(e) $\int_C \frac{\tan(z/2)}{(z-x_0)^2}\,dz = 2\pi i \sec^2(\frac{x_0}{2}) \cdot \frac{1}{2} = \pi i \sec^2(\frac{x_0}{2})$

No. 2

(a) $\int_C g(z)\,dz = \int \frac{1}{(z+2i)(z-2i)}\,dz = \frac{1}{z+2i}\Big|_{z=2i} \cdot 2\pi i = \frac{\pi}{2}$ (since $2i$ is interior to the circle)

(b) $\int_C g(z)\,dz = \int \frac{1}{(z-2i)^2(z+2i)^2}\,dz = 2\pi i \left[\frac{1}{(z+2i)^2}\right]'\Big|_{z=2i} = 2\pi i \cdot (-2) \cdot (z+2i)^{-3}\Big|_{z=2i} = \frac{\pi}{16}$

No. 3.

proof: $g(2) = \int_C \frac{2s^2-s-2}{s-2}\,ds = 2\pi i \cdot [2s^2-s-2]\big|_{s=2} = 8\pi i$ (since 2 is interior to $|z|=3$)

When $|z|>3$, $\frac{2s^2-s-2}{s-z}$ is analytic at all points interior to and on $|z|=3$.

Then by Cauchy – Goursat Thm., $g(z) = \int_C \frac{2s^2-s-2}{s-z}\,dz = 0$

No. 4.

proof: $g(z) = \int_C \frac{s^3+2s}{(s-z)^3}\,ds = \frac{2\pi i}{2!}(s^3+2s)''\big|_{s=z} = 6\pi i z$ (when z is inside C).

When z is outside C, by Cauchy – Goursat Thm. (since $\frac{s^3+2s}{(s-z)^3}$ is analytic at all points interior to and on C), we have $g(z)=0$.

No. 7

proof: $z_0 = 0$ is inside the unit circle.

Thus, $\int_C \frac{e^{az}}{z}\,dz = 2\pi i \cdot e^{az}\big|_{z=0} = 2\pi i$

On the other hand, $\int_C \frac{e^{az}}{z}\,dz = \int_{-\pi}^{\pi} \frac{e^{a(\cos\theta + i\sin\theta)}}{\cos\theta + i\sin\theta}\, i(\cos\theta + i\sin\theta)\,d\theta$

$= i\int_{-\pi}^{\pi} e^{a\cos\theta + ia\sin\theta}\,d\theta$

$= i\int_{-\pi}^{\pi} [e^{a\cos\theta}\cos(a\sin\theta) + ie^{a\cos\theta}\sin(a\sin\theta)]\,d\theta$

$= 2\pi i$

Therefore, $\int_{-\pi}^{\pi} e^{a\cos\theta}\cos(a\sin\theta)\,d\theta = 2\pi$

i.e. $\int_0^{\pi} e^{a\cos\theta}\cos(a\sin\theta)\,d\theta = \pi$

Given $z_0 \in \mathbb{C}$.

10. proof: By Cauchy's Inequality, we have

$$|f''(z_0)| \le \frac{2M_R}{R^2} \le \frac{2A|z|}{R^2} \le \frac{2A(|z_0|+R)}{R^2}$$

Taking $R \to \infty$, we get $f''(z)$ is zero everywhere.

Thus $f(z) = a_1 z + a_2$ where a_1 and a_2 are constants.

Noted $|f(0)| = |a_2| \le 0$, we have $a_2 = 0$.

Therefore, $f(z) = a_1 z$, where a_1 is a contant.

Sect. 59 No. 1

proof: Since f is entire, we have g is entire.

$$|g(z)| = |e^{f(z)}| = |e^{u(x,y)}| \le e^{u_0}$$

By Liouville's Thm. g is constant.

Thus, u is also constant.

No. 4.

proof: By maximum modulus principle, $|f(z)|$ attain its maximax on the boundary of R.

Noted $\sin x$ attain max. at $x = \frac{\pi}{2}$ in $[0, \pi]$ and $\sinh y$ attain max. at $y=1$ in $[0,1]$

Thus $f(z)$ has a maximum value at $z = \frac{\pi}{2} + i$ (since $f(\frac{\pi}{2}+i) > 0$)

No. 6.

$f(z) = e^x \cos y + i e^x \sin y$

$u(x,y) = e^x \cos y \qquad x \in [0,1],\ y \in [0, \pi]$

$u(x,y)$ attains max. at $x=1$, $y=0$, i.e $z=1$

$u(x,y)$ attain min. at $x=1$, $y=\pi$ i.e $z = 1+\pi i$

which illustrate results in SEC. 59 and Ex. 5.